ON THE MATHEMATICAL THEORY OF THE INTERNAL FRICTION AND LIMITING STRENGTH OF ROCKS UNDER CONDITIONS OF STRESS EXISTING IN THE INTERIOR OF THE EARTH

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INTRODUCTION

That solid bodies could be permanently deformed and made to flow without rupture under sufficiently great stress has long been known. The extensive experiments of Tresca on the flow of metals[1] (1864–72) directed the attention of several mathematicians of the time to the subject. Tresca announced as a result of his experiments the simple law that a stressed solid would commence to flow as soon as the maximum shearing stress exceeded a limiting value K characteristic of the solid. This hypothesis was incorporated into the elastic solid theory by Saint-Venant[2] and others. The hope was expressed by these writers that by effecting the solution of simple problems in "plasticodynamics," corresponding to the experimental arrangements employed, it might be possible, not only to verify the theoretical results, but also to determine a specific constant K characteristic of the various metals and related in an intimate manner to other physical constants. It was found possible, however, to solve only a very limited number of extremely simple problems: (1) circular cylinder under uniform pressure over the plane ends or subject to uniform lateral pressure; (2) cylindrical shell constrained to remain of constant length and subject to uniform internal and external pressure; (3) circular cylinder twisted beyond the elastic limit; (4) bar of rectangular section bent by a suitable distribution of forces to take the form of a circular arc.

[1] H. Tresca, *Par. Mém. Sav. Etr.*, XX (1872), 75 ff. and 281 ff. A summary of Tresca's experiments is given by L. S. Ware, *Journal of the Franklin Institute*, LXXIII (1877), 418 f.

[2] Saint-Venant, *Comptes Rendus*, LXVII (1868), 131 ff., 203 ff., 278 ff.; LXVIII (1869), 221 ff., 290 ff.

None of these simple problems corresponded, however, to any detailed observations available. The position with regard to the final mathematical interpretation of Tresca's observations was summed up by Saint-Venant in a communication to the French Academy.[1] It was stated that, before much progress could be made in formulating a mathematical theory of plastic flow, it would be necessary to plan experiments more easily capable of mathematical specifications; in particular he recommended that means be taken to trace out in the *interior* of the solid the extent of the plastic deformations. The difficulty of doing this without at the same time interfering with the continuity of the solid under test has apparently not been overcome up to the present, so that data on plastic deformation available for mathematical treatment are still very meager.

It is interesting to notice, however, that we have available at the present day a method of exploring the internal structure of solids which seems to fulfil the need expressed by Saint-Venant. By the use of extremely powerful X-rays it has been found possible to detect internal cavities in steel castings not visible on the surface. The subject has recently been extensively studied by Davey,[2] who states that it is possible to detect an air-inclusion 0.021 inch thick in $1\frac{1}{4}$ inches of steel and an air-inclusion 0.007 inch thick in $\frac{5}{8}$ inch of steel. More recently Pilon,[3] making use of the Coolidge tube, has successfully penetrated 5.5 centimeters of steel. This method appears to the writer to offer the means of studying in successive stages the plastic deformation of specimens of various materials under conditions of intense stress. In these circumstances it would be necessary only to drill extremely fine holes in the specimen in various directions and to study the deformation of these as the solid is made to flow.

Tresca's hypothesis that flow in a solid commences and continues as long as the shearing stress exceeds a definite limit has been found

[1] Saint-Venant, "De la suite qu'il serait nécessaire de donner aux recherches expérimentales de plasticodynamique," *Comptes Rendus*, LXXXI (juillet, 1875), 115–21.

[2] W. P. Davey, *Trans. Am. Electrochem. Soc.*, XXVIII (1915), 407–18.

[3] H. Pilon, *Rev. de Mét.*, XII (Nov., 1915), 1017–23.

by later tests to be only approximately true. It is found that to produce continuous flow in a plastic solid it is necessary continuously to increase the distorting stress. A simple illustration of this fact is to be noticed in the manner in which a short circular cylinder crushed in a testing machine ultimately breaks down. According to Tresca's theory the surfaces of shear should be cones of semi-vertical angle of 45°, while experiments indicate that the angle is more often in the neighborhood of 55° for a material like cast iron.[1] These results have led to a modification of Tresca's hypothesis as already mentioned. The effect of this so-called "resistance to flow" does not appear to have been studied with a view to formulating the laws according to which solids may be made to flow continuously.

In the field of experimental ballistics the use of the permanent deformation of short copper cylinders to measure the enormously high pressures involved in testing explosives by means of the so-called "crusher-gauge," invented by Noble about 1875,[2] has led to the detailed study of the relation of applied stress and deformation produced in these special circumstances.[3] The results of these observations have recently been studied in detail by Brillouin.[4] The behavior of copper shows the existence of internal friction analagous to that observed by Adams and Bancroft in the case of various rock specimens.

In the experiments carried out by the latter investigators the use of nickel-steel jackets of standard thickness to incase the rock specimens subjected to flow is analagous to the use of short cylinders of annealed copper in the crusher-gauges just referred to. In order to obtain the lateral pressure on the specimen corresponding to a given deformation of the nickel-steel jacket, a calibration-curve is obtained by filling the cylinders with tallow. The hydrostatic pressures required to give a series of deformations give the required

[1] A. Morley, *Strength of Materials* (Longmans, Green, & Co., 1908), p. 55.

[2] See *Encyclopaedia Britannica*, 11th ed., article on "Ballistics," for a brief description of the crusher-gauge.

[3] Vieille, *Mémorial des poudres et salpêtres* (Gauthier-Villars, Paris),V, 12–61.

[4] M. Brillouin, "Les grandes déformations du cuivre par écrasement et par traction," *Ann. de Chimie et de Physique*, 9e série, II (1914), 489–96.

calibration-curve, just as the copper cylinders of the crusher-gauge are calibrated under known end pressures in a testing machine.

MATHEMATICAL DISCUSSION OF THE OBSERVATIONS OF ADAMS AND BANCROFT DURING THE ELASTIC STAGE

Although the experiments which form the subject of the present discussion were all carried out when both rock and nickel-steel had been deformed beyond the elastic limit, it is not without interest, especially in view of further experiments on the subject, to follow out the distribution of stresses in the rock specimen and in the nickel-steel throughout the elastic stage. The necessary theory from which the formulas given below are derived has been given by the writer in a previous paper.[1] As in that discussion, it is sufficient for the present purpose to consider the ideal problem of plane stress, that is, one in which the end pressures and lateral pressures are such that the displacements at the outer surfaces, both of the rock specimen and of the nickel-steel jackets, are everywhere symmetrical with respect to the axis and everywhere constant for a given load. In reality the nickel-steel jacket shows a bulge over the center of the specimen. As long as this is not too great the analysis will give an approximate representation of the state of stress in the central portion of the specimen and nickel-steel jacket at which the measurements of displacement were taken by means of a sensitive extensometer. The justification for this mode of treatment has already been noticed in the writer's paper previously referred to in its application to a similar problem.

We denote by $\widehat{rr}$ the stress component along the radius r; by $\widehat{\theta\theta}$, the component at right angles to r; and by $\widehat{zz}$, that along the axis. According to Lamé's notation, μ is the modulus of rigidity of the rock specimen and λ one of the moduli of elasticity such that $\kappa=(\lambda+\frac{2}{3}\mu)$ is the modulus of compression. Poisson's ratio is denoted by $\sigma=\frac{1}{2}\lambda/(\lambda+\mu)$. We denote by accented symbols the corresponding elastic constants for nickel-steel. In the problem under discussion we denote by b the radius of the rock specimen and

[1] L. V. King, "On the Limiting Strength of Rocks under Conditions of Stress Existing in the Earth's Interior," *Journal of Geology*, XX (February–March, 1912), 121–26.

the interior radius of the nickel-steel jacket, and by c the exterior radius of the nickel-steel jacket. If P is the pressure per unit area applied to the end of the test specimen, we have $\widehat{zz}=-P$. The principal shearing stresses are one-half the algebraic difference of the principal stresses and are at once obtained by writing $a=0$ in the equations (13) of the writer's paper mentioned above. We then obtain

$$\left.\begin{aligned} &\text{(i)} \quad \tfrac{1}{2}\,|\widehat{rr}-\widehat{zz}|=\tfrac{1}{2}\,\frac{\sigma}{1-\sigma}P\left\{\frac{1-2\sigma}{\sigma}+\frac{\beta}{1+\beta}\right\} \\ &\text{(ii)} \quad \tfrac{1}{2}\,|\widehat{rr}-\widehat{\theta\theta}|=0 \\ &\text{(iii)} \quad \tfrac{1}{2}\,|\widehat{\theta\theta}-\widehat{zz}|=\tfrac{1}{2}\,\frac{\sigma}{1-\sigma}P\left\{\frac{1-2\sigma}{\sigma}+\frac{\beta}{1+\beta}\right\} \end{aligned}\right\} \tag{1}$$

where

$$\beta=\frac{1+\sigma}{1-\sigma}\cdot\frac{\mu}{\mu'}\left\{1+\frac{1-\sigma'}{1+\sigma'}\frac{b^2}{c^2}\right\}\cdot\frac{1}{1-b^2/c^2} \tag{2}$$

The radial displacement U at the outer surface of the rock specimen is given by

$$\frac{U}{b}=\frac{P}{2\mu}\cdot\frac{\sigma}{1+\sigma}\,\frac{\beta}{1+\beta} \tag{3}$$

Each of the principal shearing stresses (i), (ii), (iii), is associated with a family of surfaces along which the material will crack or flow. These are illustrated in Fig. 1, reproduced from the writer's paper already mentioned. It is important to notice in the present connection that the principal shearing stresses in the interior of the rock, as given by (i) and (iii), are independent of the radius r and remain equal throughout the elastic régime. It thus follows from Tresca's theory that the rock, when stressed under these ideal conditions, will commence to break down or flow *simultaneously* throughout its entire volume. The surfaces of shear which will be associated with the elastic breakdown may either be the system of cones (i) of semivertical angle 45° or the system of helicoidal surfaces (iii) of 45° pitch giving rise to the well-known Luder's lines on the curved surface of the specimen. The particular surfaces of shear which will be observed in any particular test will depend on accidental circumstances, as either system is equally likely to occur.

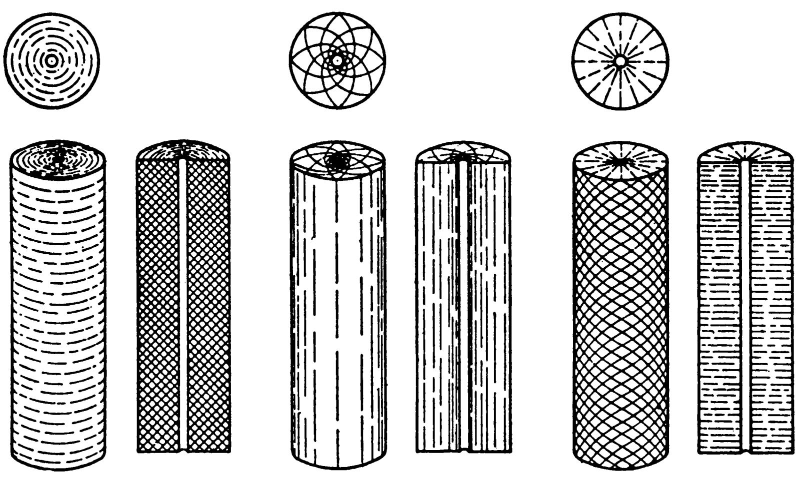
We easily derive expressions for the principal shearing stresses in the nickel-steel jacket. At points distant r' from the axis these are

$$\left.\begin{aligned}
&\text{(i)}' \quad \tfrac{1}{2}\,|\,\widehat{rr}'-\widehat{zz}'\,| = \tfrac{1}{2}P\cdot\frac{\sigma}{1-\sigma}\cdot\frac{1}{1+\beta}\,\frac{c^2/r'^2-1}{c^2/b^2-1}\\
&\text{(ii)}' \quad \tfrac{1}{2}\,|\,\widehat{rr}'-\widehat{\theta\theta}'\,| = P\frac{\sigma}{1-\sigma}\,\frac{1}{1+\beta}\,\frac{c^2/r'^2}{c^2/b^2-1}\\
&\text{(iii)}' \quad \tfrac{1}{2}\,|\,\widehat{\theta\theta}'-\widehat{zz}'\,| = \tfrac{1}{2}P\,\frac{\sigma}{1-\sigma}\,\frac{1}{1+\beta}\,\frac{c^2r'^2+1}{c^2/b^2-1}
\end{aligned}\right\} \quad (4)$$

The radial displacement U' at the outer surface of the nickel-stee jacket is given by

$$\frac{U'}{c}=\frac{P}{\mu'}\cdot\frac{b^2}{c^2}\cdot\frac{1}{1-b^2/c^2}\cdot\frac{1}{1+\sigma'} \qquad (5)$$

By writing $\mu=0$ and therefore $\beta=0$, $\sigma=\frac{1}{2}$, the foregoing give the familiar results for stresses in a cylinder subject to internal hydro-



(i) (ii) (iii)

FIG. 1.—(Figure from this Journal, Vol. XX, No. 2 [February-March, 1912], p. 123.)

static pressure. The three principal shearing stresses given above all take their maximum value (independent of sign) at the interior

surface $r=b$, and of these maxima (ii)$'$ is the greatest. The maximum shearing stress is therefore

$$\tfrac{1}{2}\,|\,\widehat{rr}'-\widehat{\theta\theta}'\,|_{max.}=P\frac{\sigma}{1-\sigma}\cdot\frac{1}{1+\beta}\cdot\frac{1}{1-b^2/c^2} \tag{6}$$

It follows from this discussion that elastic breakdown of the nickel-steel jacket commences at the interior surface and, as deformation continues, extends gradually to the outer surface. The surfaces of shear in this case are the system of cylindrical surfaces whose traces on a plane perpendicular to the axis of the cylinder are equiangular spirals intersecting orthogonally and cutting all radii at angles of 45°. An examination of the nickel-steel jackets shows, in fact, that the surfaces of shear approximated roughly to this system. The polished outer surface of stressed specimens showed indications of fine longitudinal ribs, while in such as were actually ruptured it was noticed that the surface of rupture conformed to that predicted from theory. As the rupture occurred when the nickel-steel was stressed very much beyond the elastic limit, the actual surfaces of shear are determined by very complex conditions involving the effect of internal friction, with which we shall deal in a later section.

Numerical results.—A rough verification of the preceding results may be made by calculating the relation between the load and the increase of diameter of the nickel-steel jacket according to equation (5). For nickel-steel we take $\sigma'=0.327$ and $\mu'=10.8\times10^6$ pounds per sq. in., values employed in the writer's paper just referred to. In one set of experiments (referred to as 0.25-centimeter wall) $b=1.00$ cm., $c=1.25$ cm., giving from (2)

$$\beta=3.68\times\frac{1+\sigma}{1-\sigma}\cdot\frac{\mu}{\mu'}.$$

When the jacket is filled with tallow we may take $\sigma=\tfrac{1}{2}$, $\mu=0$, $\beta=0$, so that equation (5) gives

$$U'/c=1.34\times(P/\mu'),$$

or in terms of the total load, $W=\pi b^2P$, we obtain

$$2U'\ (\text{inches})=2.52\times10^{-7}\times W\ (\text{pounds}) \tag{7}$$

When a specimen of Carrara marble is inserted we have[1] $\sigma=0.2744$, $\mu=3.154\times 10^6$ pounds per sq. in., whence $\beta=1.889$ and

$$U'/c=0.176\times(P/\mu'),\text{ or }2U'\text{ (inches)}=3.31\times 10^{-8}\times W\text{ (pounds)} \quad (8)$$

In another set of experiments (referred to as 0.33-centimeter wall), $b=1.00$ cm., $c=1.33$ cm., giving

$$\beta=2.96\times\frac{1+\sigma}{1-\sigma}\frac{\mu}{\mu'}.$$

In the case of tallow filling we find as before,

$$U'/c=0.980\times(P/\mu')\text{ or }2U'\text{ (inches)}=1.96\times 10^{-7}\times W\text{ (pounds)} \quad (9)$$

and in the case of the Carrara marble specimen

$$U'/c=0.147\times(P/\mu')\text{ or }2U'\text{ (inches)}=2.94\times 10^{-8}\times W\text{ (pounds)} \quad (10)$$

In Fig. 2 are compared the observed and theoretical stress-strain diagrams corresponding to the cases calculated out in equations (8) to (10). In the case of tallow filling, the initial slope of the observed curves agrees approximately with the calculated slope. In the case of the marble filling, the agreement is within the limits of error involved in measuring these extremely small strains.

MATHEMATICAL DISCUSSION OF THE OBSERVATIONS OF ADAMS AND BANCROFT DURING THE PLASTIC STAGE

1. *Navier's theory of internal friction.*—Let $\widehat{xx}$, $\widehat{yy}$, and $\widehat{zz}$ be the principal stresses in the solid at a point P measured toward the origin (Fig. 3). Let S be the shearing stress in a plane whose direction cosines with respect to the direction of the three principal stresses are (l, m, n). Let N be the stress normal to this plane. We then have

$$\left.\begin{aligned}&S^2+N^2=l^2\widehat{xx}^2+m^2\widehat{yy}^2+n^2\widehat{zz}^2\\ \text{and}\quad&N=l^2xx+m^2\widehat{yy}+n^2\widehat{zz}\end{aligned}\right\} \quad (11)$$

Generalizing somewhat on Navier's hypothesis of elastic breakdown, we may state that the material will not break down as long as

$$S<K \quad (12)$$

[1] Adams and Coker, "Elastic Constants of Rocks," *Publication No. 46 of the Carnegie Institution of Washington*, 1906, p. 69.

where K is a function, not only of the stress N normal to the plan at which slide occurs, but also of the previous history of the

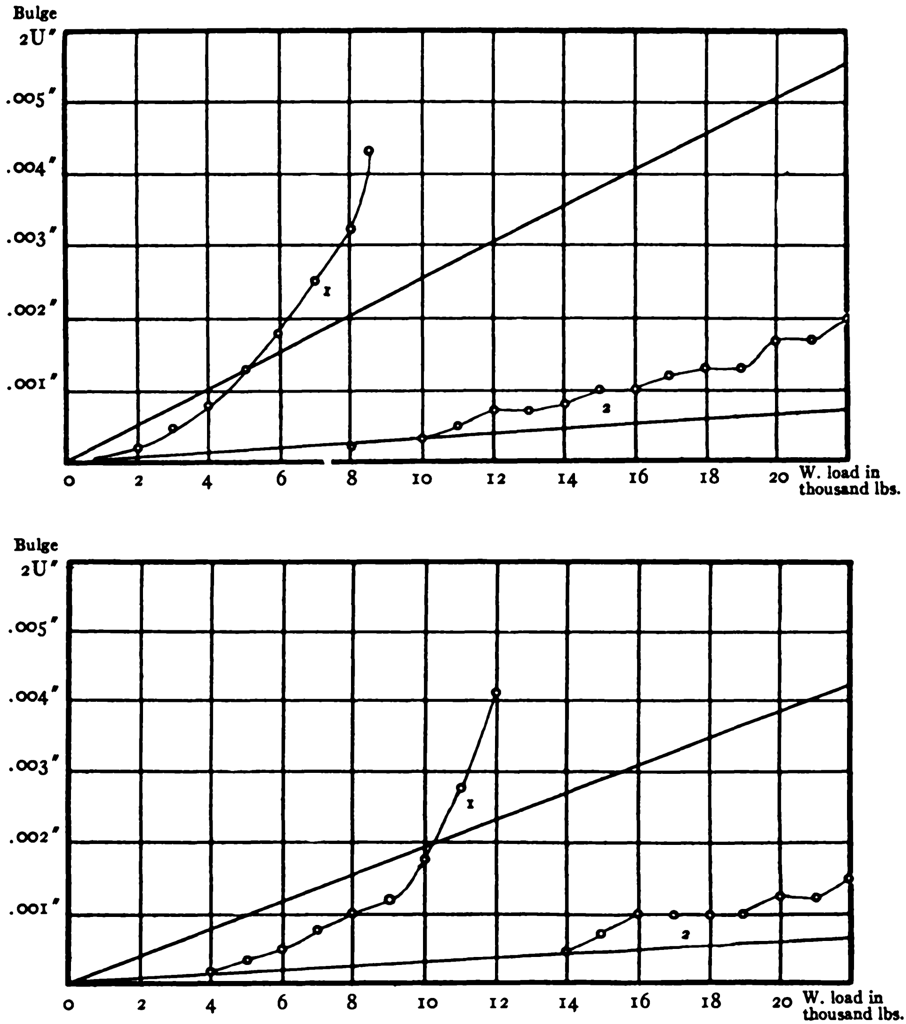


FIG. 2.—Theoretical and observed stress-strain diagrams. Curves 1, tallow filling. Curves 2, Carrara marble filling.

specimen. According to Tresca's hypothesis, K was regarded as a constant, depending only on the nature of the specimen. An exten-

sion of this hypothesis due to Navier (the so-called internal-friction theory) replaces (12) by the condition

$$S<K+\mu N,$$

μ being a new constant somewhat analogous to the coefficient of friction of mechanics. In order to discover the relation between the principal stresses at the elastic limit, it is necessary to find the direction $(l,\ m,\ n)$ which makes $(S-\mu N)$ a maximum and equate the result to K. Suppose the principal stresses to be all of the same sign, two of them equal, $\widehat{yy}=\widehat{xx}$, and $\widehat{zz}>\widehat{xx}$ (corresponding to the state of affairs in the cylindrical rock specimens under test). We then have, writing $l=\sin\theta\cos\phi$, $m=\sin\theta\sin\phi$, $n=\cos\theta$,

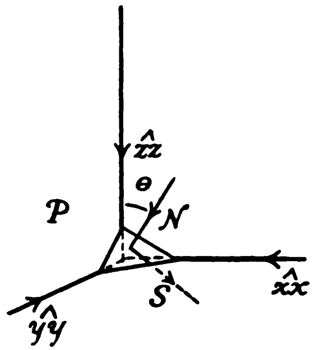


FIG. 3

$$\left.\begin{aligned} S^2+N^2=\widehat{xx}^2\sin^2\theta+\widehat{zz}^2\cos^2\theta,\quad N=\widehat{xx}\sin^2\theta+\widehat{zz}\cos^2\theta \\ S=(\widehat{zz}-\widehat{xx})\sin\theta\cos\theta \end{aligned}\right\} \tag{13}$$

$$S-\mu N=(\widehat{zz}-\widehat{xx})\sin\theta\cos\theta-\mu(\widehat{xx}\sin\theta+\widehat{zz}\cos^2\theta) \tag{14}$$

This expression reaches a maximum when

$$\cot 2\theta=-\mu, \tag{15}$$

in which circumstances

$$(S-\mu N)_{max.}=\tfrac{1}{2}(\widehat{zz}\cot\theta-\widehat{xx}\tan\theta), \tag{16}$$

and the relation between the principal stresses at breakdown is given by

$$\widehat{zz}=2K\tan\theta+\widehat{xx}\tan^2\theta \tag{17}$$

where θ is given in terms of μ (the coefficient of friction) by (15). This result indicates that the material in question will break down along a family of cones of semivertical angle $\alpha=\frac{1}{2}\pi-\theta$.

2. *Discussion of observations.*—In the experiments of Adams and Bancroft the cylindrical rock specimens were subjected to end loads transmitted by the steel pistons. As a result of the intense pressure

developed, the rock cylinders were caused to bulge out laterally over the central portion, where the thickness of the nickel-steel jacket was reduced to 0.25 centimeter and 0.33 centimeter, respectively, in the two sets of experiments. The rock was thus subjected to a continuous succession of breakdowns, so that it was possible from these observations to determine the relation between the end and lateral pressures required to keep the rock in movement.

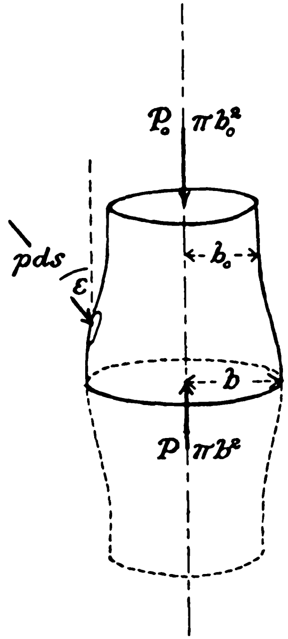


FIG. 4

Considering the central portion of the rock cylinder throughout which the flow takes place, we may reasonably assume, when the bulge is small, that the average pressure-intensity P_0 along the direction of the axis is given by

$$P_0 = W_0/(\pi b_0^2),$$

W_0 being the load on the steel piston and b_0 its radius. As the bulge becomes sensible, it is necessary to make a correction to allow for the increasing area over which the pressure is distributed. Referring to Fig. 4, we denote by P the average pressure-intensity across a plane at right angles to the axis at the position of maximum bulge where the radius of the cross section is b. We denote by p the resultant traction per unit area exerted by the nickel-steel jacket on the rock specimen in a direction making an angle ϵ with the axis. Then, considering the equilibrium of one-half of the rock specimen, we may write

$$\pi P_0 b_0^2 + \int p \cos \epsilon \, dS = \pi b^2 P, \tag{18}$$

the integral representing the total component of the tractions between the rock specimen and the nickel-steel jacket in a direction parallel to the axis of the cylinder. When an exactly similar jacket is filled with tallow and deformed by the application of a load on the steel pistons in the same way, we may consider the pressure in

the interior to be hydrostatic. If p_0 be the hydrostatic pressure required to bulge the nickel-steel jacket to the same radius b, we have instead of (18) the equation

$$\pi p_0 b_0^2 + \int p_0 \cos \epsilon_0 dS = \pi b^2 p_0, \tag{19}$$

where ϵ_0 now denotes the direction which the normal to the deformed surface makes with the axis of the cylinder. It was carefully ascertained in the experiments of Adams and Bancroft that the *shape* of the bulged nickel-steel jacket was the same when occupied by the softer rocks and such an easily flowing metal as lead, in which conditions of pressure approach very nearly to hydrostatic conditions under the very intense loads employed. As the deformation of the nickel-steel jacket is due to the distribution of surface tractions p, it is reasonable to assume that they are distributed in approximately the same way. This is equivalent to asserting that the tangential component of the surface traction between rock and nickel-steel is negligible compared to the normal component, a statement which seems to be reasonable in view of the fact that both rock and nickel-steel are highly polished over the surface of contact. We may thus write $\int p \cos \epsilon \, dS = \int p_0 \cos \epsilon_0 \, dS$ in (18) and (19) and arrive at the relation

$$P = p_0(1 - b_0^2/b^2) + P_0 b_0^2/b^2, \tag{20}$$

giving the average pressure-intensity at the center of the specimen to be identified with $\widehat{zz}$ of equation (17). The corresponding lateral pressure is given by p_0, which is identified with $\widehat{xx}$ of (17).

We are now in a position to test the theory of internal friction expressed by (17) from the observations of Adams and Bancroft. It is only necessary to plot against each other the end pressures $\widehat{zz}$ and the lateral pressures $\widehat{xx}$ as determined above. Such specimens as give straight lines may be said to possess a definite *modulus of plasticity*, K, and *coefficient of internal friction*, μ. Curves obtained in this way are shown in the Appendix, where they are described in detail for the various specimens tested. The results show that for some kinds of rock the curves approximate closely to straight lines between certain limits of pressure. In the interpretation of these curves it must be kept in mind that the material is not broken

down from an initially unstrained state[1] at each stage of the process. The constants of plasticity and internal friction, as determined by the present investigation, refer to rock which is being made to flow continuously. This state of affairs, however, approaches more nearly to that occurring in nature during slow geological deformations than to conditions existing when the rock is broken down from an initially unstrained state.

Under ideal conditions the curves for the observations taken with the nickel-steel jackets of the two wall thicknesses should be identical. Actually, however, they differ to some extent, indicating that the effect of stresses set up by the deformation of the nickel-steel has not been entirely eliminated. The two sets of observations are, however, sufficiently close to give approximate estimates of the relation between the principal stresses which must exist before the rock can be made to flow under conditions existing in the earth's crust. It will be noticed from the curves of Plate I that for the harder rocks, such as diabase and granite, the curves along which breakdown takes place show the existence of a very large coefficient of internal friction. Since the hydrostatic pressure is given by $\frac{1}{3}(2\widehat{xx}+\widehat{zz})$, this is equivalent to the statement that the *stiffness* or limiting shearing stress required to break down the rock increases with the hydrostatic pressure to which the rock is submitted. In other words, we come to the important conclusion that *the stress-difference required to break down rock material under conditions of pressure existing in the earth's crust increases with the depth.* In the application of this result to geophysical problems, the foregoing conclusion may have to be somewhat modified to take into account the rise of temperature with depth. It is highly desirable that further experiments be carried out with a view to ascertaining the influence of this factor.

NOTE ON APPLICATIONS TO GEOPHYSICAL PROBLEMS

Up to the present the only quantitative data available for use in geodynamical problems have been obtained by crushing cubes of various rocks in a testing machine according to the ordinary rules

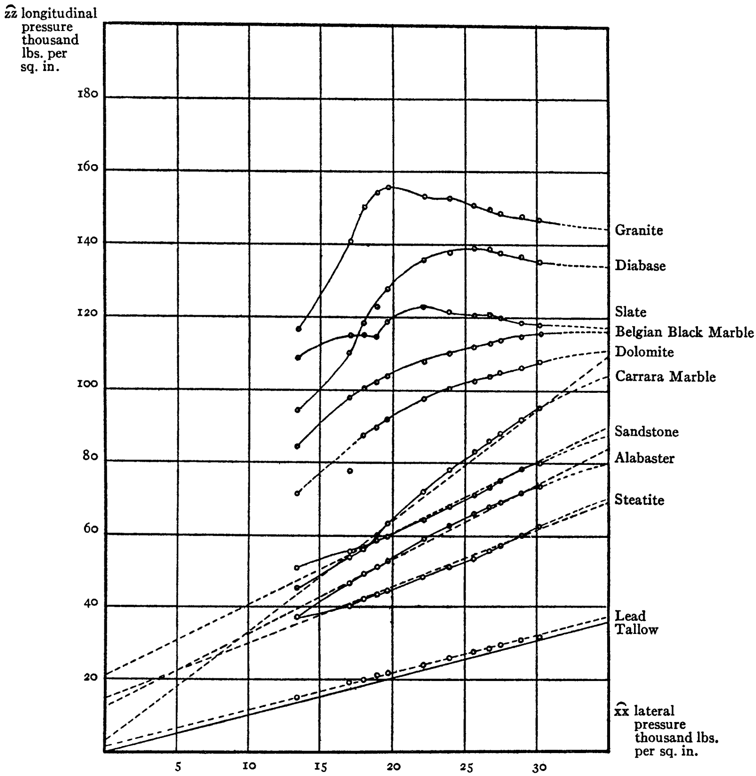
[1] Compare Karman's observations on marble and sandstone, *Zeit. des Vereins deutscher Ingenieure*, October 21, 1911.

of engineering practice. The unsatisfactory nature of such data as applied to conditions of stress deep down in the earth's crust has already been pointed out by the writer.[1] The results now available from the observations of Adams and Bancroft supply much needed data for the purposes of geophysics. Quoting from a classical paper by Sir George Darwin,[2] "With regard to the earth we require to know what is the limiting stress-difference under which a material takes permanent set or begins to flow rather than the stress-difference under which it breaks; for if the materials of the earth were to begin to flow, the continents would sink down, and the sea bottoms rise up." In the paper quoted Darwin estimates roughly the stress-difference in the interior of the earth due to a distribution of continental masses corresponding roughly to the actual distribution. For instance, it is estimated that the stress-difference under the continents of Africa and America is at a maximum at more than 1,100 miles from the earth's surface and amounts to about 4 tons per square inch. Darwin's conclusion that "marble would break under this stress, but that *strong* granite would stand" must be modified considerably in the light of the results of Adams and Bancroft, as the limiting strength of the rock material under the enormous pressure at the depth referred to would probably be increased many times. For the purposes of such calculations the curves of Plate I may be employed as they stand. If, for instance, it is desired to investigate the stability of mountain ranges or of continental elevations, the principal stresses at great depths must be derived from the theory of elasticity, making use of elastic constants derived from the interpretation of seismological records. If the principal stresses at any point be plotted as $\widehat{zz}$ and $\widehat{xx}$ on such a diagram as that of Plate I, a particular rock material will flow if the point falls between the axis $\widehat{zz}$ and the curve characteristic of the particular rock formation under consideration. The material will be on the point of flowing if the point falls on the curve itself, while the rock will stand the stress if the point falls between the

[1] L. V. King, *op. cit.*, p. 120.

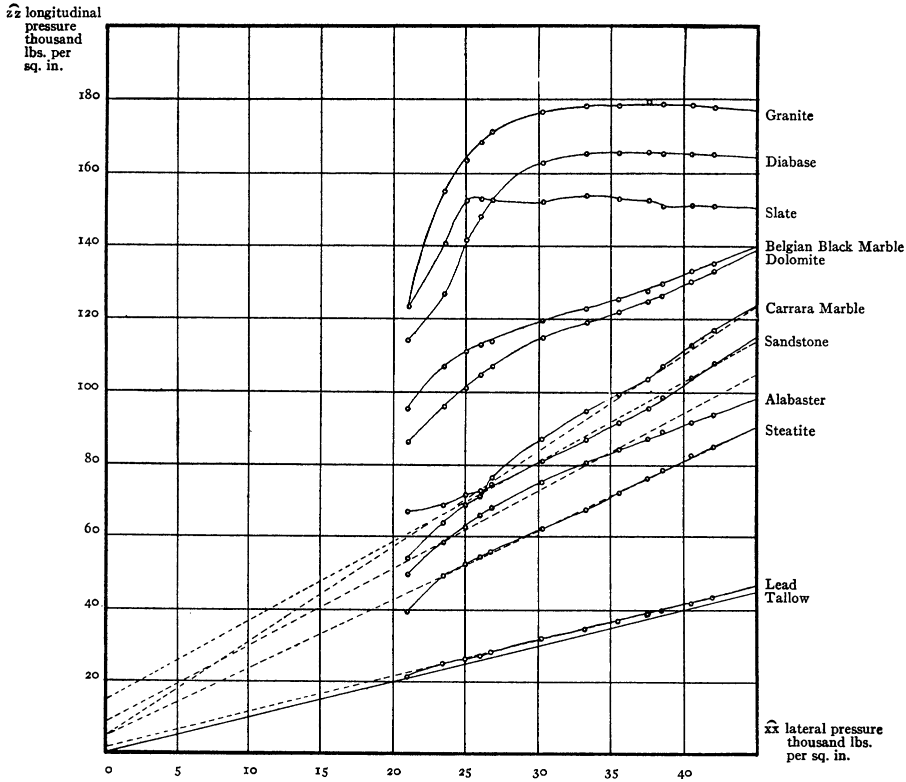
[2] Sir G. Darwin, "On the Stresses Caused in the Interior of the Earth by the Weight of Continents and Mountains," *Phil. Trans.*, CLXXIII (1882), 187–230; *Scientific Papers*, II (1908), 495.

PLATE I



Relative Plasticities of Various Rock Specimens
Nickel-steel jackets of 0.25 cm. wall

PLATE II



Relative Plasticities of Various Rock Specimens

Nickel-steel jackets of 0.33 cm. wall

curve and the axis $\widehat{xx}$. Thus for complete stability the entire series of points representing stress-differences beneath a continental elevation must fall in this latter region. It is thus evident that the existing theories of isostasy should, in considering the equilibrium of stresses called into existence by continental elevations and mountain ranges, take account of a "*compensation* of *plasticity*"—i.e., of the increased stiffness or resistance to deformation—of the underlying rock when submitted to greater hydrostatic pressure. With the reservation already made as to the possible influence of temperature, we have a considerable basis of evidence in favor of the conclusion that at any time in the past history of the earth continental elevations might have attained much greater altitudes above sea-level than any at present existing, without giving rise to stress-differences in the earth's interior sufficiently great to have caused rupture or breakdown, owing to the much increased "resistance to flow" set up in the rock material by the great pressure of the overlying crust. We should conclude also that, in the event of flow occurring, the region of flow would be confined to a region of the earth's crust comparatively near the earth's surface. The increasing limiting stress, with pressure characteristic of rock material made to flow as in Adams' and Bancroft's experiments, leads one to the conclusion that great movements of the earth's crust have for the most part always proceeded by extremely slow and continuous adjustments to pressure conditions, and not, as supposed by some geologists,[1] by a series of cataclysmal collapses of the type which would occur if the material of the earth's crust possessed in all circumstances a unique and definite limiting strength analogous to that obtained by crushing a specimen, unsupported laterally, in a testing machine. The further consideration of these problems must, however, be left over for further discussion. Enough has been said to make it evident that the results of Adams and Bancroft have provided much needed data in the light of which many of the existing theories of geodynamics may require considerable modification.

[1] G. A. J. Cole, Presidential address delivered before the geological section of the British Association, Manchester meeting, 1915, *B. A. Report*, pp. 403–20.

APPENDIX

In order to study the experimental data on the flow of rocks in the light of a theory of internal friction, the data reproduced in Tables I and II were obtained from the original large-scale curves obtained by Adams and Bancroft connecting the end load on the steel pistons with the bulge of the nickel-steel jacket. Each of the curves represented the mean of two, three, or more complete sets of observations. The first row of numbers for each specimen is the total load W_0 (in thousand pounds) on the steel pistons required to bulge the nickel-steel jacket by the amounts entered under the various columns. The second row gives the pressure-intensity $P_0 = W_0 \pi b_0^2$ in thousand pounds per square inch exerted on the end of the specimen of radius b_0. The third line gives the average pressure-intensity $P = \widehat{zz}$ in thousand pounds per square inch at the central portion of the specimen in the direction of the axis, correcting for the effect of the bulge from formula (20). It will be noticed that the average longitudinal pressure at the center is somewhat less than that over the ends by amounts which increase considerably with the harder rocks. The final results given in Tables I and II are shown graphically in Plates I and II, respectively. Against the lateral pressures (given by the experiment on tallow) are plotted the longitudinal pressures required to bulge the nickel-steel jacket to the same extent. For such of the rocks as give curves approximating to straight lines we may say that a definite *modulus of plasticity* and *coefficient of internal friction* exist. Rough estimates of these constants as determined for the soft rocks from large-scale curves are given in Table III, in which the first entry corresponds to the 0.25-centimeter wall nickel-steel jacket and the second to the 0.33-centimeter wall. It will be noticed that the two sets of results are in poor agreement for K, and are only in rough agreement for μ, the difficulty arising from attempting to fit a straight line to a series of points which are only approximately colinear.

In the case of the harder rocks no definite coefficient of friction can be said to exist. In the case of dolomite and Belgian black marble it is noticed that the coefficient of friction tends to diminish with increasing longitudinal and lateral stress. Slate gives a very irregular curve due to the development of cracks while the material is stressed. The sudden bend in the curves for diabase and granite is attributed to the actual breakdown of the rock material. From the curves of Plates I and II it will be noticed that this occurs in the neighborhood of $\widehat{zz} = 150{,}000$, $\widehat{xx} = 25{,}000$, corresponding to a stress-difference of 125,000 pounds per square inch. In a general way this result confirms the conclusion already arrived at from a discussion of the experiments of Adams on the pressure required to close up small cylindrical cavities in specimens of Westerly granite. In the writer's paper already mentioned (p. 641, n. 1) it was pointed out that the stress-difference required to break down the rock material in the neighborhood of small cavities amounted to as much as

TABLE I

0.25-Centimeter Wall		1	2	3	4	5	6	7	8	9	10	11	12
Bulge $2b_0=0.7874''$	$2b-2b_0$	0.002″	0.004″	0.006″	0.008″	0.010″	0.020″	0.030″	0.040″	0.050″	0.060″	0.080″	0.100″
	$2b$	.7894″	.7914″	.7934″	.7954″	.7974″	.8074″	.8174″	.8274″	.8374″	.8474″	.8674″	.8874″
	$1-b_0^2/b^2$	0.004	0.010	0.014	0.020	0.023	0.048	0.071	0.095	0.115	0.136	0.176	0.213
Tallow	W_0	6.5	8.2	8.8	9.2	9.6	10.8	11.7	12.5	13.0	13.4	14.1	14.7
	$P_0=\widehat{xx}$	13.4	16.9	18.1	19.0	19.7	22.2	24.0	25.7	26.7	27.5	29.0	30.3
	$\widehat{xx}=\widehat{xx}$	13.4	16.9	18.1	19.0	19.7	22.2	24.0	25.7	26.7	27.5	29.0	30.3
Steatite	W_0	18.0	19.7	20.5	21.3	21.8	24.0	25.6	27.1	28.5	29.8	32.2	34.3
	P_0	36.9	40.4	42.1	43.7	44.7	49.3	52.6	55.6	58.4	61.1	66.1	70.4
	$\widehat{xx}$	36.8	40.2	41.8	43.2	44.1	48.0	50.5	52.8	54.8	56.5	59.6	61.9
Alabaster	W_0	18.0	22.6	24.0	25.0	26.0	29.5	31.9	33.8	35.3	36.6	39.0	41.0
	P_0	37.0	46.4	49.3	51.3	53.4	60.6	65.4	69.3	72.4	75.1	80.1	84.1
	$\widehat{xx}$	36.9	46.1	48.9	50.7	52.6	58.7	62.4	65.2	67.2	68.6	71.1	72.7
Sandstone	W_0	24.5	27.0	27.7	28.7	29.5	32.2	34.5	36.5	38.4	40.0	43.0	45.3
	P_0	50.3	55.4	56.9	58.8	60.5	66.0	70.7	74.8	78.7	82.1	88.2	92.9
	$\widehat{xx}$	50.2	55.0	56.4	58.0	59.5	63.9	67.3	70.2	72.7	74.6	77.8	79.6
Marble	W_0	22.0	26.2	27.5	30.0	31.0	36.2	40.0	43.1	45.2	47.3	51.2	54.7
	P_0	45.1	53.8	56.4	61.6	63.6	74.2	82.0	88.4	92.8	97.0	105.0	112.1
	$\widehat{xx}$	45.0	53.5	55.9	60.8	62.6	71.7	77.8	82.5	85.2	87.5	91.6	94.7
Dolomite	W_0	35.0	38.0	43.0	44.2	45.5	49.2	51.7	53.7	55.4	56.9	59.7	62.5
	P_0	71.8	78.0	88.2	90.8	93.3	101.0	106.0	110.1	113.6	116.6	122.4	128.2
	$\widehat{xx}$	71.6	77.4	87.3	89.3	91.6	97.2	100.1	102.1	103.6	104.5	105.9	107.4
Belgian black marble	W_0	41.0	48.0	49.5	50.6	51.6	54.6	56.8	58.7	60.4	61.9	64.8	67.2
	P_0	84.1	98.4	101.5	103.7	105.8	112.0	116.5	120.4	123.9	127.0	132.9	137.9
	$\widehat{xx}$	83.9	97.6	100.3	101.9	103.7	107.6	109.8	111.5	112.7	113.4	11.46	115.0
Slate	W_0	(53.0)	(56.4)	(56.7)	(56.9)	58.8	62.4	62.8	63.6	64.5	65.4	67.0	69.0
	P_0	(108.7)	(115.7)	(116.3)	(116.7)	120.6	128.0	128.8	130.2	132.3	134.1	137.4	141.5
	$\widehat{xx}$	(108.3)	(114.7)	(114.9)	(114.7)	118.2	122.8	121.1	120.3	120.2	119.5	118.2	117.8
Diabase	W_0	46.0	54.0	58.5	60.7	63.5	68.9	71.5	73.3	74.5	75.5	77.7	79.7
	P_0	94.3	110.8	120.0	124.5	130.2	141.2	146.6	150.3	152.8	154.8	159.3	163.4
	$\widehat{xx}$	94.0	109.9	118.6	122.3	127.5	135.4	137.7	138.5	138.3	137.4	136.3	135.0
Granite	W_0	57.0	69.0	74.0	76.5	77.4	79.0	79.2	79.7	80.5	81.6	84.1	86.6
	P_0	116.9	142.0	151.8	156.9	158.7	159.7	162.4	163.4	165.1	167.3	172.5	177.6
	$\widehat{xx}$	116.6	140.6	149.9	154.0	155.4	153.0	152.4	150.4	149.2	148.2	147.2	146.3
Lead	W_0	7.0	9.1	9.6	10.1	10.4	11.7	12.6	13.3	13.8	14.3	14.9	15.4
	P_0	14.4	18.7	19.7	20.8	21.4	24.0	25.9	27.3	28.3	29.4	30.6	31.9
	$\widehat{xx}$	14.4	18.7	19.7	20.7	21.4	23.9	25.7	27.2	28.1	29.2	30.3	31.4

TABLE II

0.33-CENTIMETER WALL		1	2	3	4	5	6	7	8	9	10	11	12
Bulge $2b_0=0.7874''$	$2b-2b_0$	0.002 ″	0.004 ″	0.006 ″	0.008 ″	0.010 ″	0.020 ″	0.030 ″	0.040 ″	0.050 ″	0.060 ″	0.080 ″	0.100 ″
	$2b$	.7894″	.7914″	.7934″	.7954″	.7974″	.8074″	.8174″	.8274″	.8374″	.8474″	.8674″	.8874″
	$1-b_0^2/b^2$	0.004	0.010	0.014	0.020	0.023	0.048	0.071	0.095	0.115	0.136	0.176	0.213
Tallow	W_e	10.2	11.4	12.1	12.6	13.0	14.7	16.2	17.3	18.2	18.7	19.8	20.5
	$P_e=\hat{\mathfrak{R}}$	20.9	23.4	24.9	25.9	26.7	30.3	33.3	35.5	37.4	38.4	40.6	42.1
	$\hat{\mathfrak{R}}=\hat{\mathfrak{R}}$	20.9	23.4	24.9	25.9	26.7	30.3	33.3	35.5	37.4	38.4	40.6	42.1
Steatite	W_e	19.0	24.2	25.5	26.5	27.5	30.9	34.0	36.7	39.0	41.0	44.2	46.8
	P_e	39.0	49.6	52.3	54.3	56.4	63.3	69.7	75.2	80.0	84.0	90.7	95.9
	$\hat{\mathfrak{R}}$	38.9	49.3	51.9	53.7	55.6	61.7	67.1	71.5	75.1	77.7	81.8	84.4
Alabaster	W_e	24.0	28.5	30.75	32.3	33.5	37.5	40.7	43.2	45.2	47.0	49.7	52.2
	P_e	49.2	58.4	63.0	66.2	68.7	76.9	83.4	88.6	92.7	96.4	101.9	107.0
	$\hat{\mathfrak{R}}$	49.1	58.0	62.4	65.4	67.6	74.6	79.8	83.6	86.3	88.4	91.0	93.2
Sandstone	W_e	32.5	33.5	35.0	35.7	36.7	40.5	44.1	47.1	50.1	52.5	57.2	61.0
	P_e	66.6	68.7	71.8	73.3	75.2	83.0	90.4	96.6	102.7	10.76	117.3	125.0
	$\hat{\mathfrak{R}}$	66.4	68.2	71.1	72.3	74.0	80.4	86.3	90.8	95.2	98.0	103.7	107.4
Marble	W_e	26.0	31.0	33.7	35.0	37.5	43.5	48.2	51.5	54.5	57.2	62.4	66.7
	P_e	53.3	63.6	69.1	71.8	76.9	89.2	98.8	105.6	111.7	117.3	127.9	136.7
	$\hat{\mathfrak{R}}$	53.2	63.2	68.4	70.9	75.7	86.3	94.1	98.9	103.1	106.5	112.4	116.5
Dolomite	W_e	42.0	47.0	49.5	51.5	53.0	57.8	61.0	63.8	66.1	68.3	72.7	76.5
	P_e	86.1	96.4	101.5	105.6	108.7	118.5	125.0	130.8	135.6	140.0	149.2	156.9
	$\hat{\mathfrak{R}}$	85.9	95.6	100.3	104.0	106.7	114.3	118.4	121.7	124.3	126.0	130.1	132.4
Belgian black marble	W_e	46.5	52.5	54.5	55.7	56.5	60.1	62.9	65.4	67.8	70.0	74.1	77.8
	P_e	95.3	107.6	111.8	114.2	115.9	123.3	129.0	134.1	139.0	143.6	152.0	159.5
	$\hat{\mathfrak{R}}$	95.0	106.7	110.5	112.2	113.7	118.8	122.1	124.8	127.3	129.1	132.3	134.5
Slate	W_e	60.0	69.0	75.0	(75.3)	(75.5)	76.7	79.0	80.2	80.8	81.8	84.7	87.4
	P_e	123.1	141.7	154.0	(155.0)	(255.4)	157.9	162.6	164.8	166.8	168.4	174.4	179.6
	$\hat{\mathfrak{R}}$	122.6	140.4	152.1	(152.4)	(152.1)	151.7	153.3	152.5	152.0	150.5	150.7	150.3
Diabase	W_e	55.0	62.0	69.0	73.0	75.5	82.2	85.0	86.7	88.3	89.9	93.0	96.2
	P_e	114.0	127.4	141.8	150.0	155.1	168.9	175.0	178.2	181.4	184.7	191.1	197.7
	$\hat{\mathfrak{R}}$	113.6	126.3	140.1	147.5	152.0	162.2	164.8	164.7	164.9	164.7	164.6	164.6
Granite	W_e	(60.0)	76.0	80.2	83.2	85.1	89.5	92.0	94.0	96.0	97.8	101.2	104.3
	P_e	(123.1)	156.2	165.2	171.0	174.8	183.9	189.1	193.2	197.3	200.9	207.6	213.8
	$\hat{\mathfrak{R}}$	(122.6)	154.8	163.1	168.1	171.2	176.4	177.9	178.2	178.9	178.6	178.1	177.2
Lead	W_e	10.0	12.0	12.5	13.0	13.7	15.4	16.8	17.8	18.7	19.3	20.0	20.8
	P_e	20.5	24.6	25.7	26.7	28.1	31.6	34.5	36.5	38.4	39.6	41.1	42.7
	$\hat{\mathfrak{R}}$	20.5	24.5	25.6	26.7	28.0	31.5	34.4	36.4	38.3	39.4	41.0	42.7

160,000–200,000 pounds per square inch. The conclusion, there limited to small cavities, is extended by the present experiments of Adams and Bancroft to continuous rock stressed under conditions approaching those existing in

TABLE III

Specimen	K	μ
	(Pounds per square inch)	
Steatite	5,500–1,800	0.24–0.32
Alabaster	4,200–3,100	0.37–0.38
Sandstone	7,500–3,100	0.34–0.40
Marble	850–1,500	0.58–0.52
Lead	850– 500	0.00

the earth's interior, in which circumstances a limiting stress-difference several times greater than that obtained by the usual crushing test must be employed.